

THE SIZE OF THE SINGULAR SET OF A TYPE I RICCI FLOW

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ABSTRACT. In a singular Type I Ricci flow, we consider a stratification of the set where there is curvature blow-up, according to the number of the Euclidean factors split by the tangent flows. We then show that the strata are characterized roughly in terms of the decay rate of their volume, which in our context plays the role of a dimension estimate.

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1. INTRODUCTION

A one parameter family $(g(t))_{t \in [-T, 0]}$ of Riemannian metrics on a compact manifold M^n is a Ricci flow if it satisfies the evolution equation

$$(1.1) \quad \frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(g(t)).$$

If $g(t)$ can not be extended smoothly past time $t = 0$ then the flow is singular and

$$(1.2) \quad \sup_{M \times [0, T)} |\operatorname{Rm}(g(t))|_{g(t)} = \infty.$$

We call a singular Ricci flow *Type I* if there is a $C > 0$ such that

$$(1.3) \quad \sup_M |\operatorname{Rm}(g(t))|_{g(t)} \leq \frac{C}{|t|},$$

for all $t \in [-T, 0)$. For Type I flows it is shown in [Nab10] that the Cheeger-Gromov limit $(N, h(t), q)_{t \in (-\infty, 0)}$ of any blow-up sequence of the form $(M, \tau_i^{-1} g(\tau_i t), p)$, where $\tau_i \rightarrow 0$, is a gradient shrinking Ricci soliton. Namely there exists an $f \in C^\infty(N)$ such that

$$(1.4) \quad \operatorname{Ric}(h(-1)) + \operatorname{Hess}_{h(-1)} f = \frac{h(-1)}{2},$$

We will call such a limit *a tangent flow of $g(t)$ at p* .

It is proven in [EMT11] that tangent flows at singular points of the Ricci flow are necessarily non-flat. Here the set of singular points is defined as follows.

Definition 1.1. *A point $p \in M$ is at the singular set Σ of $(M, g(t))_{t \in [-T, 0]}$ if there is no neighbourhood U of p such that*

$$(1.5) \quad \sup_{U \times [-T, 0)} |\operatorname{Rm}(g(t))|_{g(t)} < \infty.$$

It is also shown that the volume of the singular set decays to zero as the flow approaches the singular time. Namely, if $\text{Vol}_{g(0)} \Sigma < \infty$ then $\lim_{t \rightarrow T} \text{Vol}_{g(t)} \Sigma = 0$. This decay motivates our work in two different ways. First of all, it raises the question whether the volume of the singular set decays to zero at a particular rate. Note that, in principle, appropriate estimates on the volume of high curvature or singular regions may lead to L^p curvature estimates along a Type I Ricci flow. From a different point of view, the decay of the volume of the singular set may be seen as a dimension estimate for Σ .

Dimension estimates for singular sets of geometric PDEs have a long history. For instance, we have dimension estimates for the singular set of mass-minimizing integral currents (see [Fed70, Alm83]), as well as for energy minimizing maps (see [SU82, Sim93]). Moreover, White in [Whi97] proves very general stratification theorems for upper-semicontinuous functions on domains in Euclidean space, which put the previous results in a general framework and make the theory applicable in a variety of contexts, including the mean curvature flow. Last but not least, one has the dimension estimates for the singular set of non-collapsed limits of Riemannian manifolds with Ricci curvature bounded below, arising from the theory of Cheeger and Colding in [CC97].

In general such stratification theorems involve decomposing the singular set Σ into an ascending sequence $\Sigma_0 \subseteq \Sigma_1 \subseteq \dots \subseteq \Sigma_N = \Sigma$ (for some $N \geq 0$) and then proving Hausdorff dimension estimates of the form $\dim \Sigma_j \leq j$. In this article, inspired particularly from [Whi97], we prove a stratification theorem for the singular set of a Type I Ricci flow.

Let Σ be the singular set of a Type I Ricci flow $(M, g(t))_{t \in [-T, 0)}$, as in Definition 1.1, and for every $j = 0, \dots, n-2$ set

$$\Sigma_j = \{x \in \Sigma, \text{ no tangent flow at } x \text{ splits as } (N^{n-j-1}, h(t)) \times (\mathbb{R}^{j+1}, g_{Eucl})\}.$$

It is clear that $\Sigma_0 \subseteq \Sigma_1 \subseteq \dots \subseteq \Sigma_{n-2} \subseteq \Sigma$.

The main result provides analogues of the Hausdorff dimension estimates above in the setting of Type I Ricci flows. However, several subtleties arise when we attempt to make sense of such estimates in the case of the Ricci flow. In the shrinking round n -sphere the singular set is n -dimensional. On the other hand, the flow shrinks the sphere to a point and we would like to regard it as developing a 0-dimensional singularity. Unfortunately, it is not known in general whether for a singular Ricci flow $(M, g(t))_{t \in [0, T)}$ (even a Type I flow) the metric spaces $(M, d_{g(t)})$ converge to a metric space (X, d_X) as t approaches the singular time T . Another way we could make sense of the concept of the singular set and its dimension would be to embed the flow in a larger ambient space, see [KL14] and [HN15]. However, this is beyond the scope of the present paper. Instead, we interpret the dimensionality of a singular stratum Σ_j via volume decay estimates, observing that along a cylindrical Ricci flow $g(t) = -2(n-j-1)t g_{S^{n-j}} + g_{Eucl}$ on $S^{n-j} \times \mathbb{R}^j$ the volume form is given by $d\mu_{g(-\tau)} = \tau^{\frac{n-j}{2}} d\mu_{g(-1)}$.

Theorem 1.1. *Fix $j = 0, \dots, n-2$ and let $\varepsilon > 0$. Then, there exist closed $A_i \subset \bar{\Sigma}_j$ ($i = 1, 2, \dots$), depending on j and ε , such that $\Sigma_j \subset \bigcup_{i=1}^{\infty} A_i$ and*

$$(1.6) \quad \frac{\text{Vol}_{g(-\tau)}(A_i)}{\tau^{\frac{n-j-\varepsilon}{2}}} \leq C(j, s, i)\tau^\beta,$$

for some $\beta = \beta(\varepsilon) \in (0, 1)$. Also, for every $\delta > 0$ there is i_0 such that

$$(1.7) \quad \text{Vol}_{g(-\tau)}(\bar{\Sigma}_j \setminus A_i) < \delta,$$

for every $i \geq i_0$ and $\tau \in (0, T]$.

Moreover, for each $x \in \Sigma_0$ there exist $R_0, \bar{\tau} > 0$ such that

$$(1.8) \quad B_g(x, -\bar{\tau}, R_0\sqrt{\bar{\tau}}) \cap \{y \in M, \Theta_g(y) \leq \Theta_g(x)\} \subseteq B_g(x, -\tau, R_0\sqrt{\tau}),$$

for every $\tau \in (0, \bar{\tau}]$.

Here, $B_g(x, t, r)$ denotes the $g(t)$ -metric ball of radius r centered at $x \in M$, $\text{Vol}_{g(t)}$ is the n -dimensional Hausdorff measure with respect to $g(t)$ and $\Theta_g(\cdot)$ is a lower semicontinuous function on M , analogous to the Gaussian density for the mean curvature flow, which is defined in Section 3.

Observe that estimate (1.8) may be interpreted as the isolatedness of points in the 0-dimensional stratum Σ_0 with a fixed density value. Of course such statement taken literally can not be true. For instance, for the shrinking round sphere S^n , $\Sigma_0 = S^n$ and all points have the same density due to symmetry. On the other hand, the diameter goes to zero and all points may be thought to represent a single singular point.

Ideally, in Theorem 1.1 we would prefer an estimate on the volume of Σ_j instead of the sets A_i . These sets arise by the decomposition of Σ_j according to the scale below which the flow is sufficiently close to a shrinking soliton. It is below that scale that our argument allows to iteratively refine a given covering, making it more efficient as the flow approaches the singular time. On the other hand, an estimate on the volume on the whole Σ_j would be more in the spirit of Minkowski content estimates. Such estimates for singular sets have recently been obtained using quantitative differentiation arguments in different contexts (see for instance [CN13a], [CN13b], [CHN13], [CHN15], [CNV15]). We intend to explore this direction further in a future paper.

Finally, the following corollary partially improves the volume decay statement in [EMT11] exploiting the fact that a shrinking Ricci soliton splitting more than $n-2$ Euclidean factors should necessarily be flat. Moreover, when the Weyl tensor remains bounded along the flow we obtain an improved volume decay estimate, as a consequence of the fact that Weyl-flat gradient shrinking Ricci solitons can split at most one Euclidean factor, which follows from [Zha09].

Corollary 1.1. *If $\Sigma = \Sigma_j$ it follows that for every $\varepsilon > 0$ there exist closed $A_i \subseteq \Sigma$, $i = 1, 2, \dots$, such that $\Sigma = \bigcup_{i=1}^{\infty} A_i$ and*

$$(1.9) \quad \text{Vol}_{g(-\tau)}(A_i) \leq C(i, \varepsilon)\tau^{\frac{n-j}{2}-\varepsilon},$$

for every $\tau \in (0, T]$. In particular we distinguish the following cases.

- (1) In general $\Sigma = \Sigma_{n-2}$ and $\text{Vol}_{g(-\tau)}(A_i) \leq C(i, \varepsilon)\tau^{1-\varepsilon}$.
- (2) Suppose that the Weyl curvature satisfies

$$\sup_{M \times [-T, 0)} |W_g|_g < \infty.$$

Then $\Sigma = \Sigma_1$ and $\text{Vol}_{g(-\tau)}(A_i) \leq C(i, \varepsilon)\tau^{\frac{n-1}{2}-\varepsilon}$.

In both cases, for every $\delta > 0$ there is i_0 such that $\text{Vol}_{g(-\tau)}(\Sigma \setminus A_i) < \delta$ for every $i \geq i_0$ and $\tau \in (0, T]$.

The outline of the paper is the following. In Section 2 we collect a few preliminary facts. Then, in Section 3 we introduce a monotone quantity which plays the role of Perelman's reduced volume based at the singular time, and its associated density function. They are both lower-semicontinuous under the Cheeger-Gromov convergence of Ricci flows, which is essential to our arguments. In Section 4, given any non-flat gradient shrinking Ricci soliton, we distinguish the set of points where the density function above achieves its minimum, called the spine. We then prove a splitting theorem (Theorem 4.1), which asserts that the soliton splits enough Euclidean factors \mathbb{R}^j , $0 \leq j \leq n-2$, so that its spine is of the form $V \times \mathbb{R}^j$, for some set V with uniformly bounded diameter. Finally, in Section 5 we prove Theorem 1.1 via a covering argument similar to [Sim93].

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2. PRELIMINARIES

In this section we collect some preliminary results on which we will rely in the rest of the paper.

2.1. Gradient shrinking Ricci solitons. A triplet (M^n, g, f) is called a gradient shrinking Ricci soliton if it satisfies the equation

$$(2.1) \quad \text{Ric}_g + \text{Hess}_g f = \frac{g}{2}.$$

Clearly, if (M, g) is complete with bounded curvature, the vector field $\nabla^g f$ is complete. It then follows from (2.1) that the shrinking Ricci soliton (M, g, f) induces a Ricci flow $h(t) = -t\phi_t^*g$ on M , where ϕ_t are the diffeomorphisms generated by $\nabla^g f$ via

$$\begin{aligned} \frac{d}{dt}\phi_t &= -\frac{1}{t}\nabla^g f \circ \phi_t, \\ \phi_{-1} &= \text{id}_M. \end{aligned}$$

It is well known that the following identity holds

$$(2.2) \quad R + |\nabla f|^2 - f = c.$$

The soliton function f is well-defined up to a linear function. However, when $c = 0$ in (2.2), we call f a *normalized soliton function*. Normalized soliton functions will be important to us mainly because of the following result from [Nab10].

Lemma 2.1 (Lemma 2.1 in [Nab10]). *Let $(M, g, f), (M', g', f')$ be normalized shrinking solitons and suppose that*

- (1) $\int_M e^{-f} d\mu_g, \int_{M'} e^{-f'} d\mu_{g'} < +\infty$,
- (2) (M, g) and (M', g') are isometric.

Then, $\int_M e^{-f} d\mu_g = \int_{M'} e^{-f'} d\mu_{g'}$.

2.2. The reduced distance and volume under Type I curvature bounds.

Definition 2.1 (Type I Ricci flow). *For every positive integer n and $C > 0$ we define the following classes of pointed complete Ricci flows*

$$\mathcal{RF}^C = \{(M^n, g(t), p)_{t \in (-T, 0)}, g(t) \text{ solves (1.1) and } |Rm_g|_g \leq \frac{C}{|t|} \text{ on } M \times (-T, 0)\}.$$

$$\mathcal{RF}_{\text{reg}}^C = \{(M^n, g(t), p) \in \mathcal{RF}^C, \sup_{M \times (-T, 0)} |Rm_g|_g < +\infty\}.$$

We may equip the space \mathcal{RF}^C with the topology of smooth (C^∞) pointed Cheeger-Gromov-Hamilton convergence for Ricci flows. Since for any $\mathfrak{g} = (M, g(t), p)_{t \in (-T, 0)}$ and $T_i > 0$ such that $T_i \searrow 0$, the sequence $\mathfrak{g}_i = (M, g(T_i + t), p)_{t \in (-T+T_i, 0)}$ converges to \mathfrak{g} , it follows that $\overline{\mathcal{RF}_{\text{reg}}^C} = \mathcal{RF}^C$.

Definition 2.2. *For $\mathfrak{g} = (M, g(t), p)_{t \in [-T, 0]} \in \mathcal{RF}_{\text{reg}}^C$, the reduced distance function, originally defined in [Per02], is given by*

$$(2.3) \quad l_{\mathfrak{g}}(x, \bar{\tau}) = \inf_{\gamma} \left\{ \frac{1}{2\sqrt{\bar{\tau}}} \int_0^{\bar{\tau}} \sqrt{\bar{\tau}} (R_{g(-\tau)}(\gamma(\tau)) + \left| \frac{d\gamma}{d\tau} \right|_{g(-\tau)}^2) d\tau \right\},$$

where $\gamma : [0, \bar{\tau}] \rightarrow M$, $\gamma(0) = p$, $\gamma(\bar{\tau}) = x$.

The following estimate from [Nab10] will be crucial to our work.

Proposition 2.1. *Let $\mathfrak{g} = (M, g(t), p) \in \mathcal{RF}_{\text{reg}}^C$. There exists $A = A(n, C) > 0$ such that*

- (1) $\frac{1}{A} \left(1 + \frac{d_{g(-\tau)}(p, x)}{\sqrt{\tau}}\right)^2 - A \leq l_{\mathfrak{g}}(x, \tau) \leq A \left(1 + \frac{d_{g(-\tau)}(p, x)}{\sqrt{\tau}}\right)^2$,
- (2) $|\nabla l_{\mathfrak{g}}|(x, \tau) \leq \frac{A}{\sqrt{\tau}} \left(1 + \frac{d_{g(-\tau)}(p, x)}{\sqrt{\tau}}\right)$,
- (3) $\left| \frac{\partial l_{\mathfrak{g}}}{\partial \tau} \right|(x, \tau) \leq \frac{A}{\tau} \left(1 + \frac{d_{g(-\tau)}(p, x)}{\sqrt{\tau}}\right)^2$.

Now, whenever $\mathfrak{g}_i \rightarrow \mathfrak{g}$, with $\mathfrak{g}_i, \mathfrak{g} \in \mathcal{RF}^C$, it is possible to limit out the reduced distance functions of \mathfrak{g}_i using the estimates in Proposition 2.1, as is done in [Nab10]. This motivates the following definition.

Definition 2.3. Let $\mathfrak{g} = (M, g(t), p)_{t \in (-T, 0)} \in \mathcal{RF}^C$. A function $l : M \times (0, T) \rightarrow \mathbb{R}$ is called a singular reduced distance if the following holds. There exists a sequence $\mathfrak{g}_i \in \mathcal{RF}_{\text{reg}}^C$ converging to \mathfrak{g} in the topology of \mathcal{RF}^C such that $l_{\mathfrak{g}_i} \rightarrow l$ in $C_{\text{loc}}^{0, \alpha}$.

Remark 2.1. Since $\overline{\mathcal{RF}_{\text{reg}}^C} = \mathcal{RF}^C$ it follows that the set of singular reduced distance functions of $\mathfrak{g} \in \mathcal{RF}^C$ is non-empty. Moreover, since the estimates of Proposition 2.1 pass to the limit, it follows that the space of singular reduced distance functions of $\mathfrak{g} \in \mathcal{RF}^C$ is compact in the $C_{\text{loc}}^{0, \alpha}$ topology.

Given a singular reduced distance l on \mathfrak{g} and $\tau \in (0, T)$, following [Per02] we define the reduced volume associated to l as

$$(2.4) \quad \mathcal{V}_{\mathfrak{g}, l}(\tau) := \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-l(\cdot, \tau)} d\mu_g.$$

Remark 2.2. Note that if $\mathfrak{g} \in \mathcal{RF}_{\text{reg}}^C$ then any singular reduced distance function l of \mathfrak{g} is given by (2.3). To see this, consider $\mathfrak{g}_i \rightarrow \mathfrak{g}$, with $\mathfrak{g}_i, \mathfrak{g} \in \mathcal{RF}_{\text{reg}}^C$. By Perelman's pseudolocality theorem (see [Per02]) it follows that \mathfrak{g}_i have uniformly bounded curvature in time intervals $[-a, 0]$, $a < T$, hence they converge to \mathfrak{g} uniformly locally in $M \times (-T, 0]$. Hence, the reduced distance functions $l_{\mathfrak{g}_i}$ pointwise converge to $l_{\mathfrak{g}}$.

In the following lemma we collect a few useful facts about $\mathcal{V}_{\mathfrak{g}, l}(\cdot)$.

Lemma 2.2. The reduced volume $\mathcal{V}_{\mathfrak{g}, l}(\tau)$ with respect to a singular reduced distance l of $\mathfrak{g} \in \mathcal{RF}^C$ is non-increasing in τ . Moreover, if there exist $0 < \tau_1 < \tau_2$ such that

$$(2.5) \quad \mathcal{V}_{\mathfrak{g}, l}(\tau_1) = \mathcal{V}_{\mathfrak{g}, l}(\tau_2),$$

then $g(t)$ is a gradient shrinking Ricci soliton, namely

$$(2.6) \quad \text{Ric}(g(-\tau)) + \text{Hess}_{g(-\tau)} l(\cdot, \tau) = \frac{1}{2\tau} g(-\tau),$$

and $l(\cdot, 1)$ is a normalized soliton function.

Proof. The monotonicity statement is essentially Lemma 2.8 in [Nab10]. The statement (2.6) and the fact that $l(\cdot, 1)$ is a normalized soliton function are proven together with Theorem 2.1 in [Nab10]. \square

2.3. The non-inflating property of the Ricci flow. Now we recall the non-inflating property of smooth compact Ricci flows, as it appears in Zhang [Zha12]. A similar result was also obtained by Chen and Wang in [CW13] under additional assumptions. The result in [Zha12] however is more suitable for the setting of Type I Ricci flows.

Theorem 2.1 (Theorem 1.1 in [Zha12]). Let $(M^n, g(t))_{t \in [0, t_0]}$ be a smooth and compact Ricci flow. Then for every $\alpha > 0$ there exists a $\kappa > 0$, depending on $n, \alpha, g(0)$, with the following property. If for some $x_0 \in M$ and $r \in (0, \sqrt{t_0})$ the estimate

$$(2.7) \quad R(g(t)) \leq \frac{\alpha}{t_0 - t},$$

holds in $B_g(x_0, t_0, r) \times [t_0 - r^2, t_0]$, then

$$(2.8) \quad \text{Vol}_{g(t_0)}(B_g(x_0, t_0, r)) \leq \kappa r^n.$$

The non-inflating property has the following consequence for Type I flows, which is a direct consequence of Theorem 2.1 and the Type I curvature bound.

Corollary 2.1. *Let $(M, g(t), x_0)_{t \in [-T, 0]} \in \mathcal{RF}^C$. Then, there exists a $\kappa_0 > 0$ depending on $n, g(-T), C$, such that for every $r \in (0, \sqrt{\frac{T}{2}})$ and $t_0 \in (-\frac{T}{2}, 0)$*

$$(2.9) \quad \text{Vol}_{g(t_0)}(B_g(x_0, t_0, r)) \leq \kappa_0 r^n.$$

Proof. Since $(M, g(t), x)_{t \in [-T, 0]} \in \mathcal{RF}^C$, there exists $c(n, C) > 0$ such that $R(g(t)) \leq \frac{c(n, C)}{|t_0|} \leq \frac{c(n, C)}{t_0 - t}$ on $M \times [t_0 - r^2, t_0]$. Estimate (2.9) then follows from Theorem 2.1. \square

3. THE REDUCED VOLUME BASED AT THE SINGULAR TIME

In this section we define a monotone quantity which plays the role of a reduced volume based at a singular time and consider the associated density function.

In particular, the compactness property of Remark 2.1 allows the following definition.

Definition 3.1. *We define the singular reduced volume at scale $\tau > 0$ of $\mathfrak{g} \in \mathcal{RF}^C$ as*

$$\mathcal{V}_{\mathfrak{g}}(\tau) = \mathcal{V}_{\mathfrak{g}, \bar{l}}(\tau),$$

where \bar{l} is the minimizer of $\mathcal{V}_{\mathfrak{g}, l}(\tau)$ among all singular reduced distances l of \mathfrak{g} .

A direct implication of this definition is the following.

Proposition 3.1. *Let $\mathfrak{g} = (M, g(t), p)_{t \in (-T, 0)} \in \mathcal{RF}^C$. Then*

- (1) *If $0 < \tau_1 < \tau_2$ then $\mathcal{V}_{\mathfrak{g}}(\tau_1) \geq \mathcal{V}_{\mathfrak{g}}(\tau_2)$.*
- (2) *If $\mathcal{V}_{\mathfrak{g}}(\tau_1) = \mathcal{V}_{\mathfrak{g}}(\tau_2)$ for some $0 < \tau_1 < \tau_2$, then there exists a singular reduced distance l of \mathfrak{g} such that $\mathcal{V}_{\mathfrak{g}}(\tau) = \mathcal{V}_{\mathfrak{g}, l}(\tau)$, for every $\tau \in (0, T)$. Moreover, l is a normalized soliton function.*
- (3) *Let $\mathfrak{g}_i = (M_i, g_i(t), p_i) \in \mathcal{RF}^C$ such that $\mathfrak{g}_i \rightarrow \mathfrak{g}$. Then for every $\tau \in (0, T)$*

$$\liminf_i \mathcal{V}_{\mathfrak{g}_i}(\tau) \geq \mathcal{V}_{\mathfrak{g}}(\tau).$$

In particular, the monotonicity property in Proposition 3.1 justifies the following definition.

Definition 3.2. *The density of $\mathfrak{g} \in \mathcal{RF}^C$ is defined as*

$$(3.1) \quad \Theta_{\mathfrak{g}} := \lim_{\tau \searrow 0} \mathcal{V}_{\mathfrak{g}}(\tau).$$

Remark 3.1. By Proposition 3.1, if $\mathfrak{g}_i, \mathfrak{g} \in \mathcal{RF}^C$ and $\mathfrak{g}_i \rightarrow \mathfrak{g}$ it follows that

$$(3.2) \quad \liminf_{i \rightarrow \infty} \Theta_{\mathfrak{g}_i} \geq \Theta_{\mathfrak{g}}.$$

Remark 3.2. For every $\mathfrak{g} \in \mathcal{RF}_{\text{reg}}^C$ the reduced volume satisfies $\mathcal{V}_{\mathfrak{g}, l_{\mathfrak{g}}}(\tau) \in (0, 1]$ for every $\tau > 0$, since $\lim_{\tau \rightarrow 0^+} \mathcal{V}_{\mathfrak{g}, l_{\mathfrak{g}}}(\tau) = 1$ (see for instance [CCG⁺07]). Thus, $\Theta_{\mathfrak{g}} \in [0, 1]$ for every $\mathfrak{g} \in \mathcal{RF}^C$.

Definition 3.3. Let $(M, g(t))_{t \in (-T, 0)}$ be a Type I Ricci flow and $x \in M$. We will always denote $\mathfrak{g}_x := (M, g(t), x)_{t \in (-T, 0)}$. Suppose that for some $C > 0$, $\mathfrak{g}_x \in \mathcal{RF}^C$. Then, the density of g at x is defined as

$$(3.3) \quad \Theta_g(x) = \Theta_{\mathfrak{g}_x}.$$

Remark 3.3. In [EMT11] the authors introduce a different notion of reduced volume based at the singular time and density function for a Type I Ricci flow. In their approach, they consider limits of reduced distance functions arising from different sequences of times converging to the singular time. Then, they define a singular reduced distance by considering the infimum over all these limits, and use this to build a monotone quantity and its associated density. It is not clear to the author if the density function in [EMT11] behaves well under Cheeger-Gromov convergence. Instead, the lower semicontinuity is essentially built in the definition of our density.

The reduced volume based at the singular time involves minimizing over all approximating sequences of Ricci flows, and in principle is hard to compute. However, for shrinking Ricci solitons, we obtain the following result.

Lemma 3.1. Suppose that $\mathfrak{g} := (M, g(t), p)_{t \in (-\infty, 0)} \in \mathcal{RF}^C$ is the Ricci flow associated to a normalized gradient shrinking Ricci soliton $(M, g(-1), f)$. Then,

- (1) Let l be an arbitrary singular reduced distance function for \mathfrak{g} . Then

$$\lim_{\tau \rightarrow \infty} \mathcal{V}_{\mathfrak{g}}(\tau) = \lim_{\tau \rightarrow \infty} \mathcal{V}_{\mathfrak{g}, l}(\tau) = \int_M (4\pi)^{-\frac{n}{2}} e^{-f} d\mu_{g(-1)}.$$

- (2) If $p \in M$ is a critical point of f , then

$$\Theta_g(p) = \int_M (4\pi)^{-\frac{n}{2}} e^{-f} d\mu_{g(-1)} \leq \Theta_g(x),$$

for every $x \in M$. In particular the density function Θ_g on a shrinking Ricci soliton attains a minimum.

- (3) If a singular reduced distance function l for \mathfrak{g} is also a soliton function, then

$$\mathcal{V}_{\mathfrak{g}}(\tau) = \mathcal{V}_{\mathfrak{g}, l}(\tau),$$

for every $\tau > 0$.

Proof. The first statement is essentially Theorem 3.2 in [Nab10]. We describe its proof again here for completeness.

Fix a $\tau > 0$, and let l be a singular reduced distance for \mathfrak{g} . Take a sequence $\tau_i \rightarrow +\infty$ and define the blow-down sequence $\mathfrak{g}_i := (M, \tau_i^{-1}g(\tau_i t), p)_{t \in (-\infty, 0)}$ and set $l_i(\cdot, \tau) = l(\cdot, \tau_i \tau)$. Note that l_i is a singular reduced distance for \mathfrak{g}_i .

By monotonicity and the scaling behaviour of l it follows that

$$(3.4) \quad \lim_{\tau \rightarrow \infty} \mathcal{V}_{\mathfrak{g}, l}(\tau) = \lim_{i \rightarrow \infty} \mathcal{V}_{\mathfrak{g}, l}(\tau_i \tau) = \lim_{i \rightarrow \infty} \mathcal{V}_{\mathfrak{g}_i, l_i}(\tau).$$

On the other hand, there exists $q \in M$ such that $\mathfrak{g}_i \rightarrow \mathfrak{g}_q = (M, g(t), q)_{t \in (-\infty, 0)}$. To prove this, observe that since $g(t) = -t\phi_t^* g(1)$

$$(3.5) \quad \tau_i^{-1} g(\tau_i t) = (\phi_t^{-1} \circ \phi_{\tau_i t})^* g(t),$$

$$(3.6) \quad = \phi_{-\tau_i}^* g(t),$$

since $\phi_t^{-1} \circ \phi_{\tau_i t} = \phi_{-\tau_i}$.

Hence, the sequence $(M, \tau_i^{-1} g(\tau_i t), p)$ is isometric to $(M, g(t), \phi_{-\tau_i}(p))$. Now, since $\phi_{-\tau_i}(p) \rightarrow_i q$, where q is a critical point of f and thus $\phi_t(q) = q$, it follows that $\mathfrak{g}_i \rightarrow \mathfrak{g}_q := (M, g(t), q)$.

Moreover, there is a singular reduced distance \bar{l} for \mathfrak{g}_q such that $l_i \rightarrow \bar{l}$. Hence, using the estimates in Lemma 2.1 we conclude that for every $\tau > 0$

$$(3.7) \quad \lim_{i \rightarrow \infty} \mathcal{V}_{\mathfrak{g}_i, l_i}(\tau) = \mathcal{V}_{\mathfrak{g}_q, \bar{l}}(\tau),$$

which together with (3.4) implies that \bar{l} is a normalized soliton function, by Lemma 2.2. Therefore, using Lemma 2.1 we obtain

$$(3.8) \quad \mathcal{V}_{\mathfrak{g}_q, \bar{l}}(\tau) = \int_M (4\pi)^{-\frac{n}{2}} e^{-f} d\mu_{g(-1)}.$$

Finally, combining (3.4), (3.7) and (3.8) we obtain that

$$(3.9) \quad \lim_{\tau \rightarrow \infty} \mathcal{V}_{\mathfrak{g}, l}(\tau) = \int_M (4\pi)^{-\frac{n}{2}} e^{-f} d\mu_{g(-1)},$$

for every singular reduced distance function l for \mathfrak{g} . This suffices to prove the first statement of the lemma.

To prove the second assertion, let $p, x \in M$, $\nabla f(p) = 0$ and denote, as usual, $\mathfrak{g}_p = (M, g(t), p)_{t \in (-\infty, 0)}$, $\mathfrak{g}_x = (M, g(t), x)_{t \in (-\infty, 0)}$. We will first prove that

$$(3.10) \quad \Theta_g(p) = \lim_{\tau \rightarrow \infty} \mathcal{V}_{\mathfrak{g}_p}(\tau) = \int_M (4\pi)^{-\frac{n}{2}} e^{-f} d\mu_{g(-1)}.$$

Consider $\tau_i \rightarrow 0$ and observe using (3.6) that $\mathfrak{g}_i = (M, g_i(t)\tau_i^{-1}g(\tau_i t), p)$ is isometric to $(M, g(t), \phi_{-\tau_i}p)$. Since p is a critical point of f , it follows that $\phi_{-\tau_i}(p) = p$, hence $\mathfrak{g}_i \rightarrow \mathfrak{g}_p$.

As in the proof of the first assertion of the lemma, let l be an arbitrary singular reduced distance for \mathfrak{g} and set $l_i(\cdot, \tau) = l(\cdot, \tau_i \tau)$.

By the monotonicity of $\mathcal{V}_{\mathfrak{g}, l}(\cdot)$ and the scaling behaviour of l , it follows that

$$(3.11) \quad \lim_{\tau \rightarrow 0} \mathcal{V}_{\mathfrak{g}, l}(\tau) = \lim_{i \rightarrow \infty} \mathcal{V}_{\mathfrak{g}, l}(\tau_i \tau) = \lim_{i \rightarrow \infty} \mathcal{V}_{\mathfrak{g}_i, l_i}(\tau).$$

Moreover, $\mathcal{V}_{\mathfrak{g}_i, l_i}(\tau) \rightarrow \mathcal{V}_{\mathfrak{g}, \bar{l}}(\tau)$, for some singular reduced distance \bar{l} for \mathfrak{g} , since $\mathfrak{g}_i \rightarrow \mathfrak{g}$.

Therefore, $\mathcal{V}_{\mathfrak{g}, \bar{l}}(\tau) = \lim_{\tau \rightarrow 0} \mathcal{V}_{\mathfrak{g}, l}(\tau)$, for every $\tau > 0$, which implies that \bar{l} is a normalized soliton function. By Lemma 2.1 it follows that $\lim_{\tau \rightarrow 0} \mathcal{V}_{\mathfrak{g}, l}(\tau) = \int_M (4\pi)^{-\frac{n}{2}} e^{-f} d\mu_g$. Since l was arbitrary, this implies (3.10).

The assertion of the lemma then follows from

$$\Theta_g(x) \geq \lim_{\tau \rightarrow \infty} \mathcal{V}_{\mathfrak{g}_x}(\tau) = \lim_{\tau \rightarrow \infty} \mathcal{V}_{\mathfrak{g}_p}(\tau) = \Theta_g(p),$$

where again we used (3.9). Note that by estimates on the growth of the soliton function of a gradient shrinking Ricci soliton (see for instance [Nab10] or [HM11]), f always has a critical point, hence Θ_g always attains a minimum.

For the last assertion, note that if l is a singular reduced distance for \mathfrak{g} which is also a soliton function, then by monotonicity we obtain, for every $\tau > 0$,

$$(3.12) \quad \mathcal{V}_{\mathfrak{g},l}(\tau) \geq \mathcal{V}_{\mathfrak{g}}(\tau) \geq \lim_{\tau \nearrow \infty} \mathcal{V}_{\mathfrak{g}}(\tau) = \lim_{\tau \nearrow \infty} \mathcal{V}_{\mathfrak{g},l}(\tau) = \mathcal{V}_{\mathfrak{g},l}(\tau).$$

Hence $\mathcal{V}_{\mathfrak{g}}(\tau) = \mathcal{V}_{\mathfrak{g},l}(\tau) = \lim_{\tau \nearrow \infty} \mathcal{V}_{\mathfrak{g}}(\tau)$. □

Given a shrinking Ricci soliton $(M, g(t))_{t \in (-\infty, 0)}$, Lemma 3.1 shows that the limit

$$(3.13) \quad \lim_{\tau \rightarrow \infty} \mathcal{V}_{\mathfrak{g}_x}(\tau)$$

does not depend on $x \in M$. This naturally leads to the following definition.

Definition 3.4. *The asymptotic reduced volume from the singular time of a shrinking Ricci soliton $(M, g(t))_{t \in (-\infty, 0)}$ is defined as*

$$(3.14) \quad \mathcal{ARV}(M, g) := \lim_{\tau \rightarrow \infty} \mathcal{V}_{\mathfrak{g}}(\tau).$$

Remark 3.4. The notion of the asymptotic reduced volume in the setting of ancient super-solutions to the Ricci flow was first introduced by Yokota in [Yok09], where the reduced volume is based at a non-singular time. However, it is not clear to the author whether the arguments in [Yok09] generalize to the setting of singular Type I flows, to show the invariance of the limit (3.13) on the basepoint. For our work though, it suffices to establish this just for shrinking Ricci solitons.

4. SPLITTING RICCI SHRINKERS.

Definition 4.1. *Let $(M, g(-1), f)$ be a gradient shrinking Ricci soliton, $g(t)$ the associated Ricci flow, and $p \in M$ a minimizer of Θ_g . The subset*

$$(4.1) \quad S(M, g) = \{x \in M, \Theta_g(x) = \Theta_g(p)\},$$

will be called the spine of the gradient shrinking Ricci soliton.

Remark 4.1. The lower semi-continuity of the density implies that $S(M, g)$ is closed.

Lemma 4.1 (Splitting principle). *Let $g(t) = g_M(t) + g_{Eucl}$, $t \in (-\infty, 0)$, be a gradient shrinking Ricci soliton on $M^k \times \mathbb{R}^{n-k}$, $0 < k \leq n$, satisfying $|Rm(g(-1))|_{g(-1)} \leq C$. Moreover, let $V \subseteq M$ such that $S(M \times \mathbb{R}^{n-k}, g) = V \times \mathbb{R}^{n-k}$. Suppose there exist $\tau > 0$ such that*

$$(4.2) \quad \frac{\text{diam}_{g_M(-\tau)}(V)}{\sqrt{\tau}} > A\sqrt{2} - 1,$$

where $A > 0$ is given by Proposition 2.1. Then, there exists a gradient shrinking Ricci soliton $(N^{k-1}, h(t))_{t \in (-\infty, 0)}$ and $V' \subseteq N$ such that $(M, g_M(t))$ splits isometrically as $(N, h(t)) \times (\mathbb{R}, g_{Eucl})$ and $S(M \times \mathbb{R}^{n-k}, g) = V' \times \mathbb{R}^{n-k+1}$.

Proof. Since $S(M \times \mathbb{R}^{n-k}, g)$ is closed, assumption (4.2) implies that there exist $x, y \in V$ satisfying

$$(4.3) \quad \frac{d_{g_M(-\tau)}(x, y)}{\sqrt{\tau}} > A\sqrt{2} - 1.$$

Let $p = (x, 0), q = (y, 0) \in S(M \times \mathbb{R}^{n-k})$. By Lemma 3.1, $\Theta_g(p) = \Theta_g(q) = \mathcal{ARV}(M \times \mathbb{R}^{n-k})$. This implies that there exist singular reduced distance functions l_p, l_q of \mathfrak{g}_p and \mathfrak{g}_q respectively, which are both soliton functions.

It follows from the soliton equation that the difference $L(\cdot, \tau) = l_p(\cdot, \tau) - l_q(\cdot, \tau)$ satisfies $\text{Hess}_{g(-\tau)} L(\cdot, \tau) = 0$. Moreover, since the metric on $M \times \mathbb{R}^{n-k}$ splits, its restriction $\bar{L} = L|_{M \times 0}$ satisfies $\text{Hess}_{g_M(-\tau)} \bar{L}(\cdot, \tau) = 0$.

We will show that $\nabla^{g_M(-\tau)} \bar{L}(\cdot, \tau) \neq 0$, which will imply a splitting $M = N \times \mathbb{R}$ and $g_M(-\tau) = \bar{h} + dr^2$. For this, observe that Proposition 2.1 gives

$$\begin{aligned} \bar{L}(x, \tau) = L(p, \tau) &\leq 2A - \frac{1}{A} \left(1 + \frac{d_{g(-\tau)}(p, q)}{\sqrt{\tau}} \right)^2, \\ \bar{L}(y, \tau) = L(q, \tau) &\geq \frac{1}{A} \left(1 + \frac{d_{g(-\tau)}(p, q)}{\sqrt{\tau}} \right)^2 - 2A. \end{aligned}$$

From (4.3) it follows that $\bar{L}(x, \tau) < 0 < \bar{L}(y, \tau)$, hence $\bar{L}(\cdot, \tau)$ is not constant. By scaling, we may assume that $\tau = 1$ and $g_M(-1) = \bar{h} + dr^2$.

Moreover, the restriction $f(\cdot)$ of $l_p(\cdot, 1)$ on $N \times 0$ satisfies

$$\text{Ric}_{\bar{h}} + \text{Hess}_{\bar{h}} f = \frac{1}{2} \bar{h},$$

hence it is a soliton function for \bar{h} .

It is easy to see there are $a, b \in \mathbb{R}$ such that $l_p((z, v), 1) = f(z) + \left(\frac{|v|}{2} + a\right)^2 + b$, for every $(z, v) \in N \times \mathbb{R}^{n-k+1}$. Hence $\nabla^{g(-1)} l_p((z, v), 1) = \nabla^{\bar{h}} f(z) + \left(\frac{|v|}{2} + a\right) \frac{\partial}{\partial r}$, denoting the radial vector field in \mathbb{R}^{n-k+1} . This implies that the diffeomorphisms ϕ_t which generate $g(t)$ may be expressed as $\phi_t(z, v) = (\phi_{1,t}(z), \phi_{2,t}(v))$, where

$$\begin{aligned} \frac{d}{dt} \phi_{1,t}(z) &= -\frac{1}{t} (\nabla^{\bar{h}} f)(\phi_{1,t}(z)), & \phi_{1,-1}(z) &= z, \\ \frac{d}{dt} \phi_{2,t}(v) &= -\frac{1}{t} \left(\frac{|\phi_{2,t}(v)|}{2} + a \right) \frac{\partial}{\partial r}, & \phi_{2,-1}(v) &= v. \end{aligned}$$

Therefore, $h(t) = -t(\phi_{1,t})^* \bar{h}$ is the Ricci flow generated by (N, \bar{h}, f) , and we obtain

$$\begin{aligned} g(t) &= -t\phi_t^* g(-1), \\ &= -t\phi_{1,t}^* h(-1) - t\phi_{2,t}^* dr^2, \\ &= h(t) + dr^2. \end{aligned}$$

Hence, the flow $(M, g(t))_{t \in (-\infty, 0)}$ splits for all time. \square

Theorem 4.1. *Let $(M^n, g(t))_{t \in (-\infty, 0)} \in \mathcal{RF}^C$ be a non-flat gradient shrinking Ricci soliton. Then, there exists an integer $2 \leq k \leq n$, a gradient shrinking Ricci soliton $(N^k, h(t))_{t \in (-\infty, 0)}$ and $D = A\sqrt{2} - 1 > 0$ (A is as in Proposition 2.1) such that*

- (1) $(M, g(t))$ splits isometrically as $(N^k, h(t)) \times (\mathbb{R}^{n-k}, g_{Eucl})$.
- (2) There is a $V \subseteq N$ such that $S(M, g) = V \times \mathbb{R}^{n-k}$ and $\text{diam}_{h(-\tau)}(V) \leq D\sqrt{\tau}$.

Proof. Let $0 \leq k \leq n$ be the minimal k with the property that $(M^n, g(t)) = (N^k, h(t)) \times (\mathbb{R}^{n-k}, g_{Eucl})$, for some non-flat gradient shrinking Ricci soliton $(N, h(t))$. Since $(M, g(t))$ is not flat, $k \geq 2$. Moreover, by the translational symmetry there exists a $V \subseteq N$ such that $S(M, g) = V \times \mathbb{R}^{n-k}$.

All we need to show is that $\text{diam}_{h(-\tau)}(V) \leq D\sqrt{\tau}$. If this is violated for some $\tau > 0$, Lemma 4.1 implies that $(N^k, h(t))$ splits a line, thus contradicting the minimality of k . \square

5. THE SIZE OF THE SINGULAR STRATA.

5.1. Density uniqueness of tangent flows. Let $\mathfrak{g} = (M, g(t), p)_{t \in [-T, 0]} \in \mathcal{RF}^C$. Given an arbitrary sequence $\tau_i \searrow 0$ consider a blow up sequence $\mathfrak{g}_i = (M, \tau_i^{-1}g(\tau_i t), p)$ converging to a tangent flow $\mathfrak{h} = (N, h(t), q) \in \mathcal{RF}^C$. As was described in the Introduction, the combined work of [Nab10], [EMT11] and [MM15] shows that \mathfrak{h} is in fact the Ricci flow induced by a gradient shrinking Ricci soliton $(N, h(-1), f)$ and is non-flat, provided that p belongs to the singular set Σ of $(M, g(t))_{t \in [-T, 0]}$ (see Definition 1.1).

A tangent flow of \mathfrak{g} may depend on the choice of the sequence τ_i , and is not unique in general. However, the following theorem asserts that all tangent flows of \mathfrak{g} should have the same asymptotic reduced volume.

Theorem 5.1. *Let $\mathfrak{g} = (M, g(t), p)_{t \in (-T, 0)} \in \mathcal{RF}^C$ and $\mathfrak{h} = (N, h(t), q)_{t \in (-\infty, 0)}$ be a tangent flow of \mathfrak{g} . Then*

$$(5.1) \quad \Theta_g(p) = \Theta_h(q) = \mathcal{ARV}(N, h).$$

Proof. The proof is similar to that of Lemma 3.1. Fix some singular reduced distance l for \mathfrak{g} and consider a sequence $\tau_i \searrow 0$ such that the blow up sequence $\mathfrak{g}_i = (M, \tau_i^{-1}g(\tau_i t), p)$ converges to $\mathfrak{h} = (N, h(t), q)_{t \in (-\infty, 0)}$. Set $l_i(\cdot, \tau) = l(\cdot, \tau_i \tau)$ for the corresponding singular reduced distance.

By Proposition 2.1 we obtain that along a subsequence l_i converge to some singular reduced distance \bar{l} for \mathfrak{h} . Moreover,

$$(5.2) \quad \mathcal{V}_{\mathfrak{h}, \bar{l}}(\tau) = \lim_{\tau \searrow 0} \mathcal{V}_{\mathfrak{g}, l}(\tau),$$

By Part 3 of Lemma 3.1 we also obtain that $\Theta_h(q) = \mathcal{V}_{\mathfrak{h},\bar{l}}(\tau)$. Hence, since $\Theta_g(p) = \liminf_i \Theta_{g_i}(p) \geq \Theta_h(q)$ we obtain

$$(5.3) \quad \Theta_g(p) \geq \Theta_h(q) = \mathcal{V}_{\mathfrak{h},\bar{l}}(\tau) = \lim_{\tau \searrow 0} \mathcal{V}_{\mathfrak{g},\bar{l}}(\tau) \geq \lim_{\tau \searrow 0} \mathcal{V}_{\mathfrak{g}}(\tau) = \Theta_g(p),$$

which proves the theorem. \square

Remark 5.1. Compare Theorem 5.1 with the entropy uniqueness of tangent flows observed by Mantegazza and Müller in [MM15]. The \mathcal{W} -entropy of a gradient shrinking Ricci soliton (N, h, f) is defined as

$$(5.4) \quad \mathcal{W} = \int_N (R_h - |\nabla f|^2 + f - n) \frac{e^{-f}}{(4\pi)^{\frac{n}{2}}} d\mu_h,$$

where R_h denotes the scalar curvature of h and f is normalized so that $\int_N \frac{e^{-f}}{(4\pi)^{\frac{n}{2}}} d\mu_h = 1$.

5.2. Estimating the size of the strata. Now we are ready to prove Theorem 1.1 and Corollary 1.1. Fix a singular compact Type I Ricci flow $(M, g(t))_{t \in [-T, 0]}$, and let $C > 0$ be such that

$$(5.5) \quad \sup_M |\text{Rm}(g(t))|_{g(t)} \leq \frac{C}{|t|},$$

for $t \in [-T, 0]$. Given any $r > 0$, set $g_r(t) = r^{-2}g(r^2t)$.

Definition 5.1. For every $x \in \Sigma$, $r > 0$ and $\delta > 0$ define

$$S^{x,r,\delta} = \{y \in B_{g_r}(x, -1, 4D), \quad \Theta_{g_r}(y) < \Theta_g(x) + \delta\}.$$

The sets $S^{x,r,\delta}$ are important because of the following lemma.

Lemma 5.1. (*Line-up lemma*) For every $\epsilon, \alpha > 0$ and $x \in \Sigma_j$, $j = 0, \dots, n-2$, there is a $\delta = \delta(x, j, \epsilon, \alpha) > 0$ such that for every $r \in (0, \delta)$ there exists a non-flat shrinking Ricci soliton $(X, z(t), m)_{t \in (-\infty, 0)} \in \mathcal{RF}^C$ with the following properties.

- (1) $(X, z(t))$ splits isometrically as $(N^{n-k}, h(t)) \times (\mathbb{R}^k, g_{\text{Eucl}})$, for some $k \leq j$.
- (2) $m \in S(X, z) = V \times \mathbb{R}^k$, where $V \subseteq N$ satisfies $\text{diam}_{h(-\tau)} V \leq D\sqrt{\tau}$, for all $\tau > 0$. Here $D = D(n, C)$ is given by Theorem 4.1.
- (3) There is a diffeomorphism $F : B_z(m, -1, 5D) \rightarrow M$, with $F(m) = x$, such that

$$F^{-1}(S^{x,r,\delta}) \subseteq \mathcal{N}_\epsilon^{z(t)}(V \times \mathbb{R}^k), \quad \text{and} \\ (1.001)^{-2}z(t) \leq F^*g_r(t) \leq 1.001^2z(t),$$

for every $t \in [-1, -\alpha]$. Here, $\mathcal{N}_\epsilon^{z(t)}(\cdot)$ denotes the ϵ -neighbourhood with respect to $z(t)$.

Proof. Fix $x \in \Sigma_j$ and $\epsilon, \alpha > 0$. Arguing by contradiction and passing to a subsequence if necessary, we obtain sequences $0 < r_i < \delta_i$ such that $\delta_i, r_i \searrow 0$ such that:

- (i) There is a non-flat shrinking Ricci soliton $(X, z(t), m)_{t \in (-\infty, 0)} \in \mathcal{RF}^C$ satisfying (1), (2) and $(M, g_{r_i}(t), x) \rightarrow (X, z(t), m)$. Moreover, there are diffeomorphisms $F_i : B_z(m, -1, 5D) \rightarrow M$, with $F_i(m) = x$ and

$$(1.001)^{-2} z(t) \leq F_i^* g_{r_i}(t) \leq 1.001^2 z(t),$$

for every $t \in [-1, -\alpha]$.

- (ii) There are sequences $t_i \in [-1, \alpha]$ and $y_i \in B_{g_{r_i}}(x, -1, 4D)$ such that $t_i \rightarrow \bar{t}$, $F_i^{-1}(y_i) \rightarrow y_\infty$ and $\Theta_{g_{r_i}}(y_i) < \Theta_g(x) + \delta_i$, but $F^{-1}(y_i) \notin \mathcal{N}_\epsilon^{z(t_i)}(V \times \mathbb{R}^k)$.

It follows that $y_\infty \notin \mathcal{N}_\epsilon^{z(\bar{t})}(S(X, z))$. However, the lower semicontinuity of the density under Cheeger-Gromov convergence and Theorem 5.1 imply that

$$(5.6) \quad \Theta_z(y_\infty) \leq \liminf_i \Theta_{g_{r_i}}(y_i) \leq \Theta_g(x) = \Theta_z(m).$$

Hence $y_\infty \in S(X, z)$, which is a contradiction. \square

Lemma 5.2. (Covering lemma) *Let $(X, z(t), m)_{t \in (-\infty, 0)} \in \mathcal{RF}^C$ be a non-flat, shrinking Ricci soliton satisfying properties (1) and (2) of Lemma 5.1 and $s = j + \varepsilon$, $j = 0, \dots, n-2$, $\varepsilon > 0$. Then, there is $\sigma_s \in (0, \frac{1}{2})$ such that for every $\sigma \in (0, \sigma_s]$, $\rho \in (0, 4D)$, $\alpha \in (0, (\frac{\sigma\rho}{2D})^2)$ and $x \in \mathbb{R}^k$ we obtain the covering*

$$\mathcal{N}_{\frac{\sigma\rho}{4}}^{z(-\alpha)}(V \times B_\rho(x)) \subseteq \bigcup_{l=1}^{P(\sigma)} B_z((q, x_l), -\alpha, \frac{\sigma\rho}{1.001^2}),$$

for some $x_l \in B_\rho(x) \subseteq \mathbb{R}^k$, $l = 1, \dots, P(\sigma)$, and $q \in V$. Moreover, $P(\sigma)$ satisfies

$$(5.7) \quad P(\sigma)\sigma^j \leq C(n),$$

$$(5.8) \quad P(\sigma)\sigma^s \leq \frac{1}{2}.$$

Proof. There is a $C(n) > 0$ such that, for every $\sigma > 0$ and $k \leq j$, we can cover the unit ball $\overline{B_1(0)} \subseteq \mathbb{R}^k$ with $P(\sigma)$ balls of radius $\frac{\sigma}{4}$ such that

$$(5.9) \quad P(\sigma)\sigma^j \leq C(n).$$

Moreover, we can chose $\sigma > 0$ small enough in order to satisfy

$$(5.10) \quad P(\sigma)\sigma^s \leq \frac{1}{2}.$$

Hence, we can cover any $B_\rho(x) \subseteq \mathbb{R}^k$ by $P(\sigma)$ balls of radius $\frac{\rho\sigma}{4}$,

$$(5.11) \quad B_\rho(x) \subseteq \bigcup_{l=1}^{P(\sigma)} B_{\frac{\rho\sigma}{4}}(x_l),$$

for some $x_l \in B_\rho(x)$, $l = 1, \dots, P(\sigma)$, so that (5.7) and (5.8) hold.

Now, suppose that $m = (q, 0) \in V \times \mathbb{R}^k$, let $y = (p, \bar{x}) \in \mathcal{N}_{\frac{\sigma\rho}{4}}^{z(-\alpha)}(V \times B_\rho(x))$ and choose $y' = (q', x') \in V \times B_\rho(x)$ such that $d_{z(-\alpha)}(y, y') < \frac{\sigma\rho}{4}$. Then, from (5.11) there is x_{l_0} such that $x' \in B_{\frac{\rho\sigma}{4}}(x_{l_0})$.

We then compute

$$(5.12) \quad d_{z(t)}(y, (q, x_{l_0})) \leq d_{z(t)}(y, y') + \sqrt{(\text{diam}_{h(t)} V)^2 + \left(\frac{\rho\sigma}{4}\right)^2},$$

$$(5.13) \quad \leq d_{z(t)}(y, y') + \sqrt{-tD^2 + \left(\frac{\rho\sigma}{4}\right)^2}.$$

Putting $-t = \alpha \leq \left(\frac{\sigma\rho}{2D}\right)^2$ we obtain

$$(5.14) \quad d_{z(-\alpha)}(y, (q, x_{l_0})) \leq \frac{\sigma\rho}{4}(1 + \sqrt{5}) < \frac{\sigma\rho}{1.001^2},$$

which suffices to prove the lemma. \square

Proof of Theorem 1.1. Fix j and $s = j + \varepsilon$, as in the statement of the theorem. Let $\sigma = \sigma_s \in (0, \frac{1}{2})$ and $D = D(n, C) > 0$ be the constants defined in Theorems 5.2 and 4.1 respectively, and set $\alpha = \sigma^2$, $\epsilon_s = \frac{2D1.001\sigma}{4}$.

Define, for each $i, m \geq 1$,

$$(5.15) \quad D_i = \left\{ x \in \Sigma_j, \delta(x, j, \epsilon_s, \alpha) \geq \frac{1}{i} \right\}$$

and

$$(5.16) \quad S^{i,m} = \left\{ x \in D_i, \Theta(x) \in \left[\frac{m-1}{i}, \frac{m}{i} \right) \right\}.$$

Since $\Theta_g(\cdot) \in [0, 1]$, by Remark 3.2, it follows that $D_i = \bigcup_{m=1}^i S^{i,m}$.

In the following, we fix $i \geq 1$. We will show that there exists $L_i > 0$ such that for every $q \geq 0$ and $\bar{\tau} \in [\alpha i^{-2}, i^{-2}]$

$$(5.17) \quad D_i \subseteq \bigcup_{l=1}^{Q_q} \overline{B_g(x_{q,l}, -\alpha^q \bar{\tau}, 2D\sqrt{\alpha^q \bar{\tau}})},$$

and $Q_q(2D\sqrt{\alpha^q \bar{\tau}})^s \leq 2^{-q} L_i$.

Assuming for now this is true, for each $\tau \in (0, 1]$ define the closed sets

$$\begin{aligned} C_i(\tau) &= \begin{cases} \bigcup_{l=1}^{Q_q} \overline{B_g(x_l, -\alpha^q \bar{\tau}, 2D\sqrt{\alpha^q \bar{\tau}})}, & \tau = \alpha^q \bar{\tau}, \bar{\tau} \in [\alpha i^{-2}, i^{-2}], q \geq 0, \\ C_i(i^{-2}), & \tau \geq i^{-2}, \end{cases} \\ C_i &= \bigcap_{\tau > 0} C_i(\tau), \end{aligned}$$

Using the property of the cover (5.17) and the non-inflating property of the Ricci flow we compute

$$\begin{aligned}
(5.18) \quad \text{Vol}_{g(-\tau)}(C_i) &\leq \text{Vol}_{g(-\tau)}(C_i(\tau)) \\
(5.19) \quad &= \text{Vol}_{g(-\alpha^q \bar{\tau})}(C_i(\alpha^q \bar{\tau})) \\
(5.20) \quad &\leq Q_q \kappa_0 (2D\sqrt{\alpha^q \bar{\tau}})^{n-s} (2D\sqrt{\alpha^q \bar{\tau}})^s \\
(5.21) \quad &\leq L_i \kappa_0 2^{-q} (2D\sqrt{\alpha^q \bar{\tau}})^{n-s}.
\end{aligned}$$

Since we can uniquely write $\tau = \alpha^q \bar{\tau}$, for some $\bar{\tau} \in [\alpha i^{-2}, i^{-2}]$, we conclude that

$$(5.22) \quad \frac{\text{Vol}_{g(-\tau)}(C_i)}{\tau^{\frac{n-s}{2}}} \leq L_i \kappa_0 \left(\frac{\sigma}{i}\right)^{2\beta} \tau^\beta,$$

where $\beta = -\frac{1}{2\log_2 \sigma}$. The theorem is then proven setting $A_i = C_i \cap \bar{\Sigma}_j$. Observe also that $\Sigma_j \subset \cup_{i=1}^\infty A_i$, since $\Sigma_j = \cup_{i=1}^\infty D_i$.

Moreover, since along Ricci flow the estimate $R \geq -\frac{n}{2(t-T)}$ is valid for all $t \in [-T, 0)$, it follows that there is a $C = C(n, T) > 0$ such that $\text{Vol}_{g(t)}(\bar{\Sigma}_j \setminus A_i) \leq C \text{Vol}_{g(-T)}(\Sigma_j \setminus A_i)$ for every i . Choosing i large enough so that $C \text{Vol}_{g(-T)}(\bar{\Sigma}_j \setminus A_i) < \delta$ suffices to prove (1.7).

Now we will prove (5.17) by induction. First, we claim that there exist $L_i^m, Q_0^m > 0$ and $p_{m,1}, \dots, p_{m,Q_0^m} \in M$ such that for every $\bar{\tau} \in [\frac{\alpha}{i^2}, \frac{1}{i^2}]$

$$(5.23) \quad S^{i,m} \subseteq \bigcup_{l=1}^{Q_0^m} B_g(p_{m,l}, -\bar{\tau}, 2D\sqrt{\bar{\tau}}),$$

and $Q_0^m (2D\sqrt{\bar{\tau}})^s \leq L_i^m$. To see this, observe that there exists $R_i > 0$ (depending also on the the Type I curvature bound C) such that for every $p \in M$ and $\bar{\tau} \in [\frac{\alpha}{i^2}, \frac{1}{i^2}]$

$$(5.24) \quad B_g(p, -\frac{1}{i^2}, R_i) \subseteq B_g(p, -\bar{\tau}, 2D\sqrt{\bar{\tau}}),$$

Hence, choosing a cover of $S^{i,m}$ by Q_0^m balls $B_g(x_l, -\frac{1}{i^2}, R_i)$, $l = 1, \dots, Q_0^m$, we immediately obtain (5.23), for $L_i^m = Q_0^m (2D\sqrt{\bar{\tau}})^s$.

Now, let $q \geq 0$, $\tau \in [\frac{\alpha^{q+1}}{i^2}, \frac{\alpha^q}{i^2}]$, and suppose there exist $p_l \in M$, $l = 1, \dots, Q_q^m$ such that

$$(5.25) \quad S^{i,m} \subseteq \bigcup_{l=1}^{Q_q^m} B(p_l, -\tau, 2D\sqrt{\tau}),$$

$B(p_l, -\tau, 2D\sqrt{\tau}) \cap S^{i,m} \neq \emptyset$ for every l and $Q_q^m (2D\sqrt{\tau})^s \leq 2^{-q} L_i^m$.

Choose any such ball $B_g(p_{l_0}, -\tau, 2D\sqrt{\tau})$ and $z \in S^{i,m} \cap B_g(p_{l_0}, -\tau, 2D\sqrt{\tau})$. Then, from the definitions of $S^{i,m}$ and $S^{z, \sqrt{\tau}, i^{-1}}$ it follows that

$$(5.26) \quad S^{i,m} \cap B_{g_{\sqrt{\tau}}}(z, -1, 4D) \subseteq S^{z, \sqrt{\tau}, i^{-1}}.$$

Hence, there exist $k \leq j$, $X = N^{n-k} \times \mathbb{R}^k$, $z(t) = h(t) + g_{Eucl}$, $V \subset N$ and F as in Lemma 5.1 such that

$$F^{-1}(S^{i,m} \cap B_{g_{\sqrt{\tau}}}(p_{l_0}, -1, 2D)) \subseteq F^{-1}(S^{i,m} \cap B_{g_{\sqrt{\tau}}}(z, -1, 4D)) \subseteq \mathcal{N}_{\epsilon_s}^{z(-\alpha)}(V \times \mathbb{R}^k).$$

Moreover, note that there is a ball $B_{2D(1.001)} \subseteq \mathbb{R}^j$ such that

$$(5.27) \quad F^{-1}(S^{i,m} \cap B_{g_{\sqrt{\tau}}}(p_{l_0}, -1, 2D)) \subseteq \mathcal{N}_{\epsilon_s}^{z(-\alpha)}(V \times B_{2D(1.001)}).$$

Hence, putting $\rho = 2D(1.001)$ in the covering Lemma 5.2 we obtain

$$(5.28) \quad F^{-1}(S^{i,m} \cap B_{g_{\sqrt{\tau}}}(p_{l_0}, -1, 2D)) \subseteq \bigcup_{a=1}^{P(\sigma)} B_z((q, y_a), -\alpha, \frac{2\sigma D}{1.001}),$$

for $y_a \in \mathbb{R}^k$ and $P(\sigma)\sigma^s \leq \frac{1}{2}$, since $\alpha = \sigma^2 < (\frac{\sigma\rho}{2D})^2$.

Thus, for $o_l \in M$, we get

$$(5.29) \quad S^{i,m} \cap B_{g_{\sqrt{\tau}}}(p_{l_0}, -1, 2D) \subseteq \bigcup_{l=1}^{P(\sigma)} B_{g_{\sqrt{\tau}}}(o_l, -\sigma^2, 2\sigma D),$$

$o_l \in M$, which implies that there exist $p'_l \in M$, $l = 1, \dots, Q_{q+1}^m$, $Q_{q+1}^m \leq Q_q^m P(\sigma)$ such that

$$(5.30) \quad S^{i,m} \subseteq \bigcup_{l=1}^{Q_{q+1}^m} B(p'_l, -\alpha\tau, 2D\sqrt{\alpha\tau}),$$

$B(p'_l, -\alpha\tau, 2D\sqrt{\alpha\tau}) \cap S^{i,m} \neq \emptyset$ for every l and $Q_{q+1}^m (2D\sqrt{\alpha\tau})^s \leq 2^{-(q+1)} L_i$. This proves (5.17) for $L = \sum_m L_i^m$ and $Q_q = \sum_m Q_q^m$.

In order to prove (1.8), take any $\epsilon, \alpha > 0$, $x \in \Sigma_0$ and set $\bar{\delta} = \delta(x, 0, \epsilon, \alpha)$. From Theorem 4.1 and Lemma 5.1, for every $\tau \in (0, \bar{\delta}^2]$ there is a non-flat shrinking Ricci soliton $(X, z(t), m)_{t \in (-\infty, 0)}$ such that $\text{diam}_{z(t)} S(X, z) \leq D\sqrt{-t}$, and a diffeomorphism $F : B_z(m, -1, 5D) \rightarrow M$ with $F(m) = x$ satisfying

$$(5.31) \quad F^{-1}(S^{x, \sqrt{\tau}, i^{-1}}) \subset \mathcal{N}_{\epsilon}^{z(t)}(S(X, z)),$$

$$(5.32) \quad (1.001)^{-2} z(t) \leq F^* g_{\sqrt{\tau}}(t) \leq 1.001^2 z(t),$$

for every $t \in [-1, -\alpha]$.

Now, take any $\tau \in (0, \bar{\delta}^2]$ and let $y' \in B_g(x, -\tau, 4D\sqrt{\tau}) \cap \{y \in M, \Theta_g(y) \leq \Theta_g(x)\}$. Since $\{y \in B_g(x, -\tau, 4D\sqrt{\tau}), \Theta_g(y) \leq \Theta_g(x)\} \subset S^{x, \sqrt{\tau}, \bar{\delta}}$ it follows from (5.31) that $F^{-1}(y') \in B_z(m, -\lambda, \sqrt{\lambda}D + \epsilon)$, for every $\lambda \in [\alpha, 1]$.

Then, by (5.32), $y' \in B_g(x, -\lambda\tau, 1.001(\sqrt{\lambda}D + \epsilon)\sqrt{\tau})$. Moreover, there is a $\bar{\lambda} \in [\alpha, 1]$ (independent of τ) such that for every $\lambda \in [\bar{\lambda}, 1]$,

$$(5.33) \quad B_g(x, -\lambda\tau, 1.001(\sqrt{\lambda}D + \epsilon)\sqrt{\tau}) \subset B_g(x, -\lambda\tau, 4D\sqrt{\lambda\tau}).$$

We conclude that for every $\tau \in (0, \bar{\delta}^2]$ and $\lambda \in [\bar{\lambda}, 1]$

$$(5.34) \quad B_g(x, -\tau, 4D\sqrt{\tau}) \cap \{y \in M, \Theta_g(y) \leq \Theta_g(x)\} \subset B_g(x, -\lambda\tau, 4D\sqrt{\lambda\tau}),$$

which suffices to prove (1.8) for $\bar{\tau} = \bar{\delta}^2$ and $R_0 = 4D$. \square

Below, we finish by proving Corollary 1.1.

Proof Corollary 1.1. First of all, notice that Σ is closed, hence $\overline{\Sigma_j} = \Sigma$. Now, estimate (1.9) follows from Theorem 1.1 by setting $s = j + 2\varepsilon$.

In general $\Sigma = \Sigma_{n-2}$, since every shrinking Ricci soliton which splits more than $n - 2$ Euclidean factor is flat. Therefore, the volume estimate of Case 1 follows by setting $j = n - 2$.

In Case 2, every tangent flow should have vanishing Weyl curvature. Thus, it should be isometric either to flat \mathbb{R}^n (the Gaussian soliton), or to quotients of $S^{n-1} \times \mathbb{R}$ or S^n , by [Zha09]. Hence $\Sigma = \Sigma_1$ and the volume estimate follows by setting $s = 1 + 2\varepsilon$. \square

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